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Asymptotic Stability of Some Quasilinear Parabolic Equations in Divergence Form. L^∞ Results

MIN MING TANG

Department of Mathematics, University of Missouri–Rolla, Rolla, Missouri 65401

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In this paper we study the behavior of solutions of some quasilinear parabolic equations of the form

$$(\partial u/\partial t) - \sum_{i=1}^{n} (d/dx_i)\, a_i(x, t, u, u_x) + a(x, t, u, u_x)u + f(x, t) = 0,$$

as $t \to \infty$. In particular, the solutions of these equations will decay to zero as $t \to \infty$ in the L^∞ norm.

INTRODUCTION

The behavior of solutions of nonlinear parabolic equations as $t \to \infty$ has been studied by many people. For example, Friedman [5, 6] studied the asymptotic stability of some semilinear parabolic equations. Rudenko [10] and Zelenyak [11] studied the stability of some quasilinear parabolic equations in one space variable. Levine [8] studied the nonstability of some nonlinear parabolic equations. Batra [2, 3] and Tang [9] obtained L^2 estimates in the space variables as $t \to \infty$ for some quasilinear parabolic equations.

For the equations of the type we consider, Aronson and Serrin [1] and Da Veiga [4] have proved that the solutions in the L^∞ norm had at most exponential growths. Under stronger hypotheses which involve certain algebraic sign conditions on the coefficients of the equation, we are able to prove that the solution goes to zero in the L^∞ norm as $t \to \infty$. These sign conditions enable us to show that the solution goes to zero in the L^2 norm in the space variables and that the solution satisfies a DiGiorgi–Nash type of inequality. We then employ a recursion formula to obtain an L^∞ estimate.

We assume a solution exists for the equations we study. Existence and uniqueness theorems of quasilinear parabolic equations given in divergence form can be found in the book of Ladyzhenskaya, Solonnikov, and Ural'tseva [7].

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1. Preliminaries

Notation. Let Ω be a bounded open connected domain in R^n, $n \geqslant 2$. S is the boundary of Ω, which is a smooth $(n-1)$-dimensional manifold. A point x in R^n will be denoted by $x = (x_1, \ldots, x_n)$. We define the cylinders $Q_T = \Omega \times (0, T)$, $Q_{(t_1, t_2)} = \Omega \times (t_1, t_2)$, and $Q_{\rho,(t_1,t_2)} = \{(x, t) \mid | x - x_0 | \leqslant \rho; t_1 < t < t_2\}$. We define the lateral surface of Q_T and $Q_{(t_1,t_2)}$ by S_T and $S_{(t_1,t_2)}$ respectively. mes A, $A \subset R^n$, is the standard Lebesque measure of the set A in R^n.

Function spaces. We will consider the following real Banach spaces of measurable functions with the norms

$$L^q(\Omega) = \left\{u(x) \mid \| u \|_{q,\Omega} = \left(\int_\Omega | u(x)|^q \, dx\right)^{1/q} < \infty\right\};$$

$$L^{q,r}(Q_T) = \left\{u(x, t) \mid \| u \|_{q,r,Q_T} = \left(\int_0^T \left(\int_\Omega | u(x, t)|^q \, dx\right)^{r/q} dt\right)^{1/r} < \infty\right\};$$

$$L^\infty(\Omega) = \left\{u(x) \mid \| u \|_{L^\infty(\Omega)} = \operatorname*{ess\,max}_{x\in\Omega} | u(x)| < \infty\right\};$$

$$L^\infty(Q_{(t_1,t_2)}) = \left\{u(x, t) \mid \| u \|_{L^\infty(Q_{(t_1,t_2)})} = \operatorname*{ess\,max}_{(x,t)\in Q_{(t_1,t_2)}} | u(x, t)| < \infty\right\};$$

$$W_q{}^l(\Omega) = \left\{u(x) \mid \| u \|_{W_q{}^l(\Omega)} = \sum_{j=0}^{l} \| D_x{}^j u \|_{q,\Omega} < \infty\right\};$$

$$W_1^{1,0}(Q_T) = \left\{u(x, t) \mid \| u \|_{W_2^{1,0}(Q_T)} = \left(\int_{Q_T} u^2 + | \nabla u |^2 \, dx\, dt\right)^{1/2} < \infty\right\};$$

$$W_2^{1,1}(Q_T) = \left\{u(x, t) \mid \| u \|_{W_2^{1,1}(Q_T)} = \left(\int_{Q_T} (u^2 + | \nabla u |^2 + | u_t |^2) \, dx\, dt\right)^{1/2} < \infty\right\}$$

$$V_2(Q_T) = \left\{u(x, t) \mid | u |_{Q_T} = \operatorname*{ess\,max}_{0\leqslant t\leqslant T} \| u(x, t)\|_{2,\Omega} + \| u_x \|_{2,Q_T} < \infty\right\};$$

$$V_2^{1,0}(Q_T) = \{u(x, t) \mid | u |_{V_2^{1,0}(Q_T)} = \max_{0\leqslant t\leqslant T} \| u(x, t)\|_{2,\Omega} + \| u_x \|_{2,Q_T} < \infty\};$$

$$V_2{}^0(Q_T) = V_2(Q_T) \cap \{u(x, t) \mid u = 0 \text{ on } S_T\};$$

$$\mathring{W}_m{}^l(\Omega) = W_m{}^l(\Omega) \cap \{u(x) \mid u = 0 \text{ on } S\}.$$

Inequalities. We shall need the following inequalities.

Young's inequality:

$$ab \leqslant (1/m)\, \epsilon^m \, | a |^m + ((m-1)/m)\, \epsilon^{m/(m-1)} \, | b |^{m/(m-1)},$$

where a, b are real numbers.

Holder's inequality:

$$\int_{\Omega} u(x)\, v(x)\, dx \leqslant \| u(x)\|_{q,\Omega} \| v(x)\|_{(q/(q-1)),\Omega} .$$

Embedding theorems. The following embedding theorems from [7, p. 62, p. 75] will be used.

Let $u(x) \in \mathring{W}_m^{\,1}(\Omega)$, *then*

$$\| u \|_{q,\Omega} < \beta \| u_x \|_{m,\Omega}^{\alpha} \| u \|_{r,\Omega}^{1-\alpha},$$

where (1.1)

$$\alpha = ((1/r) - (1/q))\,((1/n) - (1/m) - (1/r))^{-1}.$$

Also,

$$\| u \|_{2p/(p-2),\Omega} \leqslant c \| u_x \|_{2,\Omega} \qquad \text{for } p \geqslant n \tag{1.2}$$

where $n \geqslant 3$, *and* $p > 2$ *where* $n = 2$. *Furthermore,*

$$\| u \|_{2,\Omega} < c(\operatorname{mes} \Omega)^{1/n} \| \nabla u \|_{2,\Omega} . \tag{1.3}$$

Let $u(x, t) \in V_2^{\,0}(Q_T)$, *then*

$$\| u \|_{q,r,Q_T} < \beta \mid u \mid_{Q_T} \tag{1.4}$$

where $r \in [2, \infty]$, $q \in [2, 2n/(n-2)]$ *for* $n > 2$, *and* $r \in (2, \infty]$, $q \in [2, \infty)$ *for* $n = 2$.

For $u(x, t) \in L^{q,r}(Q_T)$, *we have*

$$\| u \|_{q_1,r_1,Q_T} \leqslant \| u \|_{q,r,Q_T} \left[\int_0^T \operatorname{mes}^{(r/q)} \tilde{A}_0(t)\, dt \right]^{(1/r)-(1/r_1)}, \tag{1.5}$$

where

$$\tilde{A}_0(t) = \{x \mid u(x, t) > 0\}, \qquad q_1 \in [1, q], \quad r_1 \in [1, r],$$
$$r_1/r = q_1/q, \qquad \text{and} \qquad q, r \geqslant 1.$$

We shall make use of the following lemmas found in [7, pp. 82–83].

LEMMA. *Let* $u(x, t) \in V_2^{1,0}(Q_T)$; *then* $u^k(x, t)$ *and* $u^{-k}(x, t) \in V_2^{1,0}(Q_T)$, $u^k(x, t) = \max_{(x,t)\in Q_T}\{u(x, t) - k, 0\}$, *and* $u^{-k}(x, t) = \max_{(x,t)\in Q_T}\{-(u(x, t) + k), 0\}$ *respectively.*

LEMMA. *Let* $u(x, t) \in V_2^{1,0}(Q_T)$; *then* $u_h(x, t)$, *the Steklov average of* $u(x, t)$, *belongs to* $W_2^{1,1}(Q_{T-\delta})$, $\delta \leqslant h$, *where* $u_h(x, t) = (1/h) \int_{t-h}^{t} u(x, \tau)\, d\tau$.

2. Main Results

Consider the equation

$$u_t - \sum_{i=1}^{n} (d/dx_i)\, a_i(x, t, u, u_x) + a(x, t, u, u_x)\, u + f(x, t) = 0, \tag{2.1}$$

for $x \in \Omega$, $t \geqslant 0$. $u(x, t) = 0$ for $x \in S$, and $u(x, 0) = \phi(x)$, $x \in \Omega$. Let the coefficients of Eq. (2.1) satisfy the hypotheses

$$a_i(x, t, u, u_x)\, u_{x_i} \geqslant \mu \mid \nabla u \mid^2, \; \mu > 0; \tag{2.1.1}$$

$$\mid a_i(x, t, u, p)\mid \leqslant \mid b(x, t)\mid \mid u \mid + \mid c(x, t)\mid \mid p \mid; \tag{2.1.2}$$

$$a(x, t, z, p) \geqslant 0 \tag{2.1.3}$$

for any $x \in \Omega$, $t > 0$ and any $z, p \in R \times R^n$;

$$\| \tfrac{1}{2} f^2(x, t)\|_{q,r,Q_{(t-T,t)}} \leqslant M \quad \text{and} \quad \| b(x, t)\|_{q,r,Q_{(t-T,t)}} \leqslant M; \tag{2.1.4}$$

where $1/r + n/2q = 1 - \chi_1$, $q \in [n/2(1 - \chi_1), \infty]$, $r \in [1/(1 - \chi_1), \infty]$, and $0 < \chi_1 < 1$;

$$\| c^2(x, t)\|_{L^\infty(Q_{(t-T,t)})} \leqslant M;$$

$$\| f(x, t)\|_{2,\Omega} \to 0 \quad \text{as} \quad t \to \infty; \tag{2.1.5}$$

$$\phi(x) \in L^2(\Omega). \tag{2.1.6}$$

Theorem 2.2. *Let $u(x, t)$ be a $V_2^{1,0}(Q_t)$ solution of* (2.1), *whose coefficients satisfy hypotheses* (2.1.1), (2.1.3), (2.1.4), (2.1.5), *and* (2.1.6). *Moreover, assume*

$$(d/dt)\, \| u(x, t)\|_{2,\Omega}^2 \text{ is a continuous function of } t. \tag{2.2.1}$$

Then the following estimate holds.

$$\| u(x, t)\|_{2,\Omega}^2 \leqslant \beta(t) \overset{\text{DEF}}{=} \| \phi(x)\|_{2,\Omega}^2\, e^{-\epsilon t} + c \int_0^t e^{-\epsilon(t-\tau)} \int_\Omega \mid f(x, \tau)\mid^2 dx\, d\tau, \tag{2.2.2}$$

where $\epsilon > 0$. Also, $\beta(t) \to 0$ as $t \to \infty$.

Theorem 2.3. *Let $u(x, t)$ be a $V_2^{1,0}(Q_t)$ ($\forall t > 0$) solution of* (2.1) *whose coefficients satisfy hypotheses* (2.1.1–2.1.6). *Suppose assumption* (2.2.1) *holds, then*

$$\| u(x, t)\|_{L^\infty(Q_{(t,\infty)})} \leqslant c(\beta(t - \tfrac{3}{4}\tau)^{\nu/2}, \tag{2.3.1}$$

where $\nu \in (0, 2(1 + \chi)/(\bar{r} + 2(1 + \chi)))$ for t and τ sufficiently large and small, respectively. In particular, $u(x, t) \to 0$, as $t \to \infty$.

Remark 2.4. The hypothesis (2.1.3) is necessary in some sense. Consider the equations

$$\begin{aligned}
u_t &= u_{xx} + u_{yy} + 4u && 0 < x < \pi, \quad 0 < y < \pi, \quad t > 0, \\
u(x, 0, t) &= u(x, \pi, t) = 0 && 0 < x < \pi, \quad t > 0, \\
u(0, y, t) &= u(\pi, y, t) = 0 && 0 < y < \pi, \quad t > 0, \\
u(x, y, 0) &= f(x, y) && 0 < x < \pi, \quad 0 < y < \pi.
\end{aligned} \tag{2.4.0}$$

The solution of (2.4.) is given by

$$u(x, y, t) = \sum_{n,m=1}^{\infty} f_{nm} e^{(4-(n^2+m^2))t} \sin nx \sin my, \tag{2.4.1}$$

where $f_{nm} = (4/\pi^2) \int_0^\pi \int_0^\pi f(x, y) \sin nx \sin my \, dx \, dy$. If $f_{11} \neq 0$, then $|u(x, y, t)| \to \infty$ as $t \to \infty$.

3. Proof of Main Theorems

Proof of Theorem 2.2. We multiply Eq. 2.1 by u_h and integrate over Ω and obtain

$$\int_\Omega u_t u_h \, dx - \int_\Omega ((d/dx_i)\, a_i(x, t, u, u_x))\, u_h \, dx = -\int_\Omega a(x, t, u, x_x)\, u u_h \, dx - \int_\Omega f(x, t)\, u_h \, dx.$$

Now

$$\int_\Omega u_t u_h \, dx = \tfrac{1}{2}(\partial/\partial t) \int_\Omega u_h{}^2 \, dx,$$

and

$$\int_\Omega -((d/dx_i)\, a_i(x, t, u, u_x))\, u_h \, dx = \int_\Omega a_i(x, t, u, u_x)\, u_{hx_i} \, dx.$$

Letting $h \to 0$, we obtain

$$\tfrac{1}{2}(\partial/\partial t) \int_\Omega u^2 \, dx = -\int_\Omega a_i(x, t, u, u_x)\, u_{x_i} + a(x, t, u, u_x)\, u^2 + f(x, t)\, u \, dx.$$

By hypotheses (2.1.1) and (2.1.3), we obtain the inequality

$$\tfrac{1}{2}(\partial/\partial t) \int_\Omega u^2 \, dx \leqslant -\mu \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |f(x, t)|\, |u| \, dx.$$

By inequality (1.3), we have

$$-\mu \int |\nabla u|^2 \, dx \leqslant (-\mu/c^2 (\text{mes}\, \Omega)^{2/n}) \int u^2 \, dx.$$

By Young's inequality,

$$\int_\Omega |f(x, t)|\,|u|\,dx \leqslant (1/2\epsilon_1)\int_\Omega |f(x,t)|^2\,dx + \tfrac{1}{2}\epsilon_1 \int_\Omega u^2\,dx.$$

Setting $\epsilon_1 = (\mu/c^2(\text{mes}\,\Omega)^{2/n})$, we obtain

$$(\partial/\partial t)\int_\Omega u^2\,dx + (\mu/c^2(\text{mes}\,\Omega)^{2/n})\int_\Omega u^2\,dx \leqslant (1/\epsilon_1)\int_\Omega |f(x,t)|^2\,dx.$$

Setting $y(t) = \| u(x,t)\|^2_{2,\Omega}$, we obtain the differential inequality of the form

$$y'(t) + \epsilon y(t) \leqslant c\int_\Omega |f(x,t)|^2\,dx.$$

Multiplying the above equation by $e^{\epsilon t}$ and integrating from $(0, t)$ and noting that $y(0) = \| \phi(x)\|^2_{2,\Omega}$, we obtain

$$0 \leqslant \| u(x,t)\|^2_{2,\Omega} \leqslant \| \phi(x)\|^2_{2,\Omega}\, e^{-\epsilon t} + ce^{-\epsilon t}\int_0^t\int_\Omega |f(x,\tau)|^2\, e^{\epsilon\tau}\,dx\,d\tau.$$

Since $\int_\Omega |f(x,t)|^2\,dx \to 0$ as $t \to 0$, $\beta(t) \to 0$.

In order to prove Theorem 2.3, we will need some local estimates for small cylinders intersecting Q_T. These estimates will be similar to the DiGiorgi–Nash type of inequalities.

THEOREM 3.1. *Let $u(x, t)$ be a $V_2^{1,0}(Q_{t_0})$ solution of Eq.* (2.1) *satisfying hypothese* (2.1.1–2.1.5). *Let*

$$Q_{\rho-\sigma_1\rho,\tau-\sigma_2\tau} = \{|\, x - x_0\,| < \rho - \sigma_1\rho,\ t_0 - (1-\sigma_2\tau) < t < t_0\}.$$

Then the following inequalities are satisfied.

$$\begin{aligned}&|\, u^k\,|^2_{Q_{\rho-\sigma_1\rho,\tau-\sigma_2\tau}\cap Q_{t_0}} \\ &\quad\leqslant \gamma\{(\sigma_1\rho)^{-2} + (\sigma_2\tau)^{-1})\,\|\, u^k\,\|^2_{2,Q_{\rho,\tau}} + (k^2+1)\,\mu^{(2(1+x)/\bar r)}(k,\rho,\tau)\}\end{aligned} \tag{3.0.0}$$

and

$$\begin{aligned}&u^{-k}\,|^2_{Q_{\rho-\sigma_1\rho,\tau-\sigma_2\tau}\cap Q_{t_0}} \\ &\quad\leqslant \gamma\{(\sigma_1\rho)^{-2} + (\sigma_2\tau)^{-1}\,\|\, u^{-k}\,\|^2_{2,Q_{\rho,\tau}} + (k^2+1)\,\mu^{(2(1+x)/\bar r)}(\bar k,\rho,\tau)\}\end{aligned} \tag{3.0.0$'$}$$

for arbitrary $k \geqslant 0$ and τ sufficiently small.

$$\mu(k,\rho,\tau) = \int_{t_0-\tau}^{t_0} \text{mes}^{(\bar r/\bar q)}\, A_{k,\rho}(t)\,dt,$$

where

$$A_{k,\rho}(t) = \{x \mid |x - x_0| \leqslant \rho, u(x, t) > k\}.$$

$$\mu(\bar{k}, \rho, \tau) = \int_{t_0-\tau}^{t_0} \operatorname{mes}^{(\bar{r}/\bar{q})} A_{\bar{k},\rho}(t)\, dt,$$

where

$$A_{\bar{k},\rho}(t) = \{x \mid |x - x_0| \leqslant \rho, -u(x, t) > k\}.$$

Proof of Theorem 3.1. Let $\xi(x, t)$ be a smooth function such that

$$0 \leqslant \xi(x, t) \leqslant 1 \quad \text{on} \quad Q_{\rho,\tau} \cap Q_{t_0}, \tag{3.1.0}$$

$$\xi(x, t) \equiv 1 \quad \text{on} \quad Q_{\rho-\sigma_1\tau, \tau-\sigma_2\tau}, \tag{3.1.1}$$

$$\xi(x, t) = 0 \quad \text{for } |x - x_0| \geqslant \rho \quad \text{and} \quad t < t_0 - \tau, \tag{3.1.2}$$

$$|\xi_x(x, t)| \leqslant c/\sigma_1\rho, \tag{3.1.3}$$

and

$$|\xi_t(x, t)| \leqslant c/\sigma_2\tau. \tag{3.1.4}$$

We multiply Eq. (2.1) by $u_h{}^k\xi^2$ and integrate over Q_T, $T \leqslant t_0$. Since,

$$\begin{aligned}\int_{t_0-\tau}^{T}\int_\Omega u_t u_h{}^k \xi^2\, dx\, dt \\ = \tfrac{1}{2}\int_\Omega (u_h{}^k(x, \tau)\, \xi(x, \tau))^2 \Big|_{t_0-\tau}^{T} dx - \int_{t_0-\tau}^{T}\int_\Omega (u_h{}^k)^2\, \xi\xi_t\, dx\, dt,\end{aligned} \tag{3.1.5}$$

and

$$\begin{aligned}\int_{t_0-\tau}^{T}\int_\Omega &-((d/dx_i)\, a_i(x, t, u, u_x))\, u_h{}^k\xi^2\, dx\, dt \\ &= \int_{t_0-\tau}^{T}\int_\Omega a_i(x, t, u, u_x)\, u^k_{hx_i}\xi^2\, dx\, dt \\ &\quad + \int_{t_0-\tau}^{T}\int_\Omega a_i(x, t, u, u_x)\, u_h{}^k 2\xi\xi_{x_i}\, dx\, dt,\end{aligned} \tag{3.1.6}$$

we obtain the identity, after letting $h \to 0$ and noting $\xi(x, t_0 - \tau) = 0$,

$$\begin{aligned}\tfrac{1}{2}\int_\Omega (u^k(x, T)\, \xi(x, T))^2\, dx + \int_{t_0-\tau}^{T}\int_\Omega a_i(x, t, u, u_x)\, u^k_{x_i}\xi^2\, dx\, dt \\ = \int_{t_0-\tau}^{T}\int_\Omega [-a_i(x, t, u, u_x)\, u^k 2\xi\xi_{x_i} + (u^k)^2\, \xi\xi_t - a(x, t, u, u_x)\, uu^k\xi^2 \\ - f(x, t)\, u^k\xi^2]\, dx\, dt.\end{aligned} \tag{3.1.7}$$

By hypotheses (2.1.1–2.1.3), we can obtain the inequality

$$\tfrac{1}{2}\int_\Omega (u^k(x,T)\,\xi(x,T))^2\,dx + \mu\int_{t_0-\tau}^{T}\int_\Omega |\nabla u^k|^2\,\xi^2\,dx\,dt$$
$$\leqslant \int_{t_0-\tau}^{T}\int [b(x,t)\,|u|\,|u^k|\,2\xi\,|\nabla\xi| + c(x,t)\,|\nabla u|\,u^k 2\xi\,|\nabla\xi| \qquad (3.1.8)$$
$$+ (u^k)^2\,\xi\,|\xi_t| + |u^k|\,|f(x,t)|\,\xi^2]\,dx\,dt.$$

Now,

$$\int_{t_0-\tau}^{T}\int_\Omega |c(x,t)|\,|\nabla u|\,|u^k|\,2\xi\,|\nabla\xi|\,dx\,dt$$
$$\leqslant \epsilon_2\int_{t_0-\tau}^{T}\int_\Omega (\nabla u^k\xi)^2\,dx\,dt + (1/\epsilon_2)\int_{t_0-\tau}^{T}\int_\Omega c^2(x,t)\,(u^k\nabla\xi)^2\,dx\,dt$$
$$\leqslant \epsilon_2\int_{t_0-\tau}^{T}\int_\Omega (\nabla u^k\xi)^2\,dx\,dt \qquad (3.1.9)$$
$$+ (1/\epsilon_2)\,|c^2(x,t)|_{L^\infty(Q_{(t_0-\tau,t_0)})}\int_{t_0-\tau}^{T}\int_\Omega (u^k\nabla\xi)^2\,dx\,dt.$$

$$\int_{t_0-\tau}^{T}\int_\Omega |b(x,t)|\,|u|\,u^k 2\xi\,|\nabla\xi|\,dx\,dt$$
$$\leqslant \int_{t_0-\tau}^{T}\int_{A_{k,\rho}(t)} |b^2(x,t)|\,|u|^2\,\xi^2\,dx\,dt + \int_{t_0-\tau}^{T}\int_\Omega (u^k\nabla\xi)^2\,dx\,dt. \qquad (3.1.10)$$

Now,

$$\int_{t_0-\tau}^{T}\int_{A_{k,\rho}(t)} |b^2(x,t)|\,|u^2|\,\xi^2\,dx\,dt$$
$$\leqslant \int_{t_0-\tau}^{T}\int_{A_{k,\rho}(t)} |b^2(x,t)|\,[(u-k)^2+k^2]\,\xi^2\,dx\,dt \qquad (3.1.11)$$
$$\leqslant \|b^2\|_{q,r,Q_{(t_0-\tau,T)}}\,[\|(u^k\xi)\|^2_{\bar q,\bar r,Q_{(t_0-\tau,T)}} + k^2\,\|1^2\|_{\bar q,\bar r,Q^{(k)}_{\rho,(t_0-\tau,T)}}],$$

where

$$Q^{(k)}_{\rho,(t_0-\tau,T)} = \{(x,t)\mid u(x,t)>k,\ t_0-\tau<t<T,\ |x-x_0|\leqslant\rho\}.$$

Now,

$$\|u^k\xi\|^2_{\bar q,\bar r,Q_{(t_0-\tau,T)}} \leqslant \mu^{(2/r)-(2/\hat r)}(k,\rho,\tau)\,\|u^k\xi\|^2_{\hat q,\hat r,Q_{(t_0-\tau,T)}}, \qquad (3.1.22)$$

by Holder's inequality, where $\bar q = 2q/(q-1)$, $\bar r = 2r/(r-1)$, $\hat q = q(1+\chi)$, $\hat r = r(1+\chi)$, $\chi = 2\chi_1/n$. Now, $(1/\hat r) + (n/2\hat q) = (n/4)$. So we may apply (1.4) to obtain

$$\|u^k\xi\|^2_{\bar q,\bar r,Q_{(t_0-\tau,T)}} \leqslant \mu^{2\chi/\hat r}(k,\rho,\tau)\,\beta^2\,|u^k\xi|_{Q_{t_0-\tau,T}}. \qquad (3.1.13)$$

Similarly,

$$\| 1^2 \|_{\bar{q},\bar{r},Q_{(t_0-\tau,T)}} \leqslant \left(\int_{t_0-\tau}^{T} \operatorname{mes}^{\bar{q}/\bar{r}} A_k(t)\, dt\right)^{2/\bar{r}} = \mu^{2(1+\varkappa)/\hat{r}}(k, \rho, \tau). \tag{3.1.14}$$

$$\int_{t_0-\tau}^{T}\int_{\Omega} | u^k | \, | f(x, t)| \, \xi^2 \, dx \, dt \leqslant \int_{t_0-\tau}^{T}\int_{\Omega} (u^k\xi)^2/2 \, dx \, dt + \int_{t_0-\tau}^{T}\int_{\Omega} (f^2(x, t)/2) \, dx \, dt. \tag{3.1.15}$$

We can show, as in the previous case, that

$$\int_{t_0-\tau}^{T}\int_{\Omega} (u^k\xi)^2/2 \, dx \, dt \leqslant \| \tfrac{1}{2} \|_{q,r,Q_{(t_0-\tau,T)}} \, | u^k\xi |^2_{Q_{t_0-\tau,T}} \, \beta^2\mu^{2\varkappa/\hat{r}}(k, \rho, \tau), \tag{3.1.16}$$

and

$$\int_{t_0-\tau}^{T}\int_{\Omega} | f^2(x, t)|^2/2 \leqslant \| f^2/2 \|_{q,r,Q_{(t_0-\tau,T)}} \, \mu^{(2/\hat{r})(1+\varkappa)}(k, \rho, \tau). \tag{3.1.17}$$

Noting that

$$\begin{aligned} | u^k\xi |^2_{Q_{(t_0-\tau,T)}} &= \operatorname*{ess\,max}_{t_0-\tau\leqslant t\leqslant T} \| u^k(x, t) \, \xi(x, t)\|^2_{2,\Omega} + \| \nabla(u^k\xi)\|^2_{2,Q_{(t_0-\tau,T)}} \\ &\leqslant \operatorname*{ess\,max}_{t_0-\tau\leqslant t\leqslant T} \| u^k(x, t) \, \xi(x, t)\|^2_{2,\Omega} \\ &\quad + \|(\nabla u^k) \, \xi \|^2_{2,Q_{(t_0-\tau,T)}} + \| u^k\nabla\xi \|_{2,Q_{(t_0-\tau,T)}} . \end{aligned} \tag{3.1.18}$$

We finally arrive at the inequality

$$\begin{aligned} &\tfrac{1}{2} \| u^k(x, T) \, \xi(x, T)\|^2_{2,\Omega} + \nu \|(\nabla u^k) \, \xi \|^2_{2,Q_{(t_0-\tau,T)}} \\ &\quad\leqslant \hat{\nu}\mu^{2\varkappa/\hat{r}}(k, \rho, \tau) \operatorname*{ess\,max}_{t_0-\tau\leqslant t\leqslant T} \| u^k(x, t) \, \xi(x, t)\|^2_{2,\Omega} \\ &\qquad + (\epsilon_2 + \hat{\nu}\mu^{2\varkappa/\hat{r}}(k, \rho, \tau)) \|(\nabla u^k) \, \xi \|^2_{2,Q_{t_0-\tau,T}} \\ &\qquad + [(1/\epsilon_2) \, | \, c^2(x, t)|_{L^\infty(Q_{(t_0-\tau,t_0)})} + \hat{\nu}\mu^{2\varkappa/\hat{r}}(k, \rho, \tau)] \, \| u^k\nabla\xi \|^2_{2,Q_{(t_0-\tau,T)}} \\ &\qquad + \|(u^k)^2 \, \xi_t \|_{1,Q_{(t_0-\tau,T)}} \\ &\qquad + (\| f^2/2 \|_{q,r,Q_{(t_0-\tau,T)}} + \| b^2 \|_{q,r,Q_{(t_0-\tau,T)}} k^2) \, \mu^{(2/\hat{r})(1+\varkappa)}, \end{aligned} \tag{3.1.19}$$

where

$$\hat{\nu} = (\| b^2 \|_{q,r,Q_{(t_0-\tau,T)}} + \| \tfrac{1}{2} \|_{q,r,Q_{(t_0-\tau,T)}}) \, \beta^2.$$

Since $\mu^{2\chi/\hat{r}}(k, \rho, \tau) \leqslant \tau^{2\chi/\hat{r}}(w_n \rho)^{2n\chi/\hat{r}}$, we may choose τ sufficiently small so that

$$\hat{\nu}\mu^{2\chi/\hat{r}}(k, \rho, \tau) \leqslant \min(\tfrac{1}{4}, \nu/4).$$

We choose $\epsilon_2 = \nu/4$. Noting (3.1.0–3.1.5), we may obtain the desired inequality (3.0.0). In order to obtain (3.0.0′), we multiply Eq. (2.1) by $\bar{u}^k \xi^2$ and note that $a(x, t, u, u_x)\, u\bar{u}^k \xi^2 \leqslant 0$. We estimate as before to obtain the desired result.

The following lemma from [7, p. 96] will be used.

LEMMA 3.2. *Suppose two sequences of nonnegative numbers y_h and z_h, $h = 0, 1, 2,\ldots$ are connected by the system of recursion inequalities*

$$y_{h+1} \leqslant cb^h(y_h^{1+\delta} + z_h^{1+\epsilon} y_h^{\delta}), \tag{3.2.0}$$

$$z_{h+1} \leqslant cb^h(y_h + z_h^{1+\epsilon}), \tag{3.2.1}$$

where $c, b, \epsilon, \delta > 0$, $b \geqslant 0$. Then

$$y_h \leqslant \lambda b^{-h/d}, \qquad z_h \leqslant (\lambda b^{-h/d})^{1/(1+\epsilon)},$$

where

$$\begin{aligned} d &= \min\{\delta, \epsilon/(1+\epsilon)\}, \\ \lambda &= \min\{(2c)^{-1/\delta}\, b^{-1/\epsilon d}, (2c)^{-(1+\epsilon)/\epsilon}\, b^{-1/\epsilon d}\}, \end{aligned} \tag{3.2.2}$$

as long as

$$y_0 \leqslant \lambda \qquad \text{and} \qquad z_0 \leqslant \lambda^{1/(1+\epsilon)}.$$

LEMMA 3.3. *Let $u(x, t)$ satisfy inequalities* (3.0.0) *and* (3.0.0′). *Let $k_h = k + k(1 - (\frac{1}{2})^h)$, $\rho_h = (\frac{1}{2} + (1/2^{h+2}))\,\rho$, and $\tau_h = (\frac{1}{2} + (1/2^{h+2})\,\tau$. Then the quantities*

$$y_h = \int_{t_0-\tau_h}^{t_0} \int_{A_{k_h,\rho_h}(t)} (u - k_h)^2\, dx\, dt, \tag{3.3.0}$$

and

$$z_h = \mu^{2/\hat{r}}(k_h, \rho_h, \tau_h) \tag{3.3.1}$$

are connected by the recursion relationships

$$y_{h+1} \leqslant \gamma_1 2^{4h}((2/k^{2\delta}) + k^{2-2\delta})\,(1 + (1/\tau)) \times (1/\rho^2)\,(y_h^{1+\delta} + z_h^{1+\chi} y_h^{\delta}), \tag{3.3.2}$$

and

$$z_{h+1} \leqslant \gamma_1 2^{4h}((2/k^{2\delta}) + k^{2-2\delta})\,(1 + (1/\tau))\,(1/\rho^2)\,(y_h + z_h^{1+\chi}). \tag{3.3.3}$$

Similarly

$$\hat{y}_h = \int_{t_0-\tau_h}^{t_0} \int_{A\bar{k}_h,\rho_h(t)} (u + k_h)^2\, dx\, dt \tag{3.3.0'}$$

and

$$\hat{z}_h = \mu^{2/\bar{r}}(k_h, \rho_h, \tau_h) \tag{3.3.1'}$$

are connected by the recursion relationships

$$\hat{y}_{h+1} \leqslant \gamma_1 2^{4h}((2/k^{2\delta}) + k^{2-2\delta})(1 + (1/\tau)) \times (1/\rho^2)(\hat{y}_h^{1+\delta} + \hat{z}_h^{1+\varkappa}\hat{y}_h{}^{\delta}), \tag{3.3.2'}$$

and

$$\hat{z}_{h+1} \leqslant \gamma_1 2^{4h}((2/k^{2\delta}) + k^{2-2\delta})(1 + (1/\tau))(1/\rho^2)(\hat{y}_h + \hat{z}_h^{1+\varkappa}). \tag{3.3.3'}$$

Proof of Lemma 3.3. Let $|x| = |x - x_0|$. Let $\xi_h(x)$ be a smooth function such that $\xi_h(x) = 1$ for $x \leqslant \rho/2^{h+1}$, $\xi_h(x) = 0$ for $|x| \geqslant \frac{1}{2}(\rho_h + \rho_{h+1}) = \bar{\rho}_h$, and $\xi_h(|x|)$ linear on $[\bar{\rho}_{h+1}, \bar{\rho}_h]$, such that $|\xi_{hx}| \leqslant 2^{h+1}/\rho$. Define

$$\lambda_h = \int_{t_0-\tau_{h+1}}^{t_0} \operatorname{mes} A_{k_{h+1},\rho_h}(t)\, dt.$$

Now

$$\begin{aligned}
y_{h+1} &\leqslant \int_{t_0-\tau_{h+1}}^{t_0} \int_{A_{k_{h+1},\rho_h}(t)} (u - k_{h+1})^2\, \xi_h{}^2(x)\, dx\, dt \\
&\leqslant \beta^2 \lambda_h^{2/(n+2)} \,|\, u^{k_{h+1}} \xi_h \,|^2_{Q_{\bar{\rho}_h,(t_0-\tau_{h+1},t_0)}} \\
&\leqslant \beta^2 \lambda_h^{2/(n+2)} \left\{ \operatorname*{ess\,max}_{t_0-\tau_{h+1}<t<t_0} \int_{A_{k_{h+1},\bar{\rho}_h}(t)} (u - k_{h+1})^2\, \xi_h{}^2\, dx \right\} \\
&\qquad + 2 \int_{t_0-\tau_{h+1}}^{t_0} \int_{A_{k_{h+1},\bar{\rho}_h}(t)} [u_x{}^2 \xi_h{}^2 + (u - k_{h+1})^2\, \xi_{hx}^2]\, dx\, dt \\
&\leqslant 2\beta^2 \lambda_h^{2/(n+2)} [\,|\, u^{k_{h+1}} \,|^2_{Q_{\bar{\rho}_h,(t_0-\tau_{h+1},t_0)}} + (4^{(h+4)}/\rho^2)\, y_h].
\end{aligned}$$

Applying inequality (3.1.0) to the function $u^{k_{h+1}}$ and the two cylinders $Q_{\bar{\rho}_h,(t_0-\tau_{h+1},t_0)}$ and $Q_{\rho_h,(t_0-\tau_h,t_0)}$, and noting that

$$\lambda_h \leqslant (k_{h+1} - k_h)^{-2}\, y_h,$$

we obtain

$$\begin{aligned}
y_{h+1} \leqslant 2\beta^2 [(k_{h+1} - k_h)^{-4/(n+2)}\, y_h^{2/(n+2)} \{\gamma[4^{h+4}\rho^{-2} + 2^{h+2}\tau^{-1}]\, \|\, u^{k_{h+1}} \,\|^2_{2,Q_{\rho_h,(t_0-\tau,t_0)}} \\
+ (k_{h+1}^2 + 1)\, \mu_h^{(2/\bar{r})(1+\varkappa)}(k_{h+1}, \rho_h, \tau_h)\} + (4^{(h+4)}/\rho^2)\, y_h].
\end{aligned}$$

Noting that $k_{h+1} - k_h = (k/2^{h+1})$, and $k_{h+1} \leqslant 2k$, we can then obtain the desired inequality (3.3.2).

Similarly we obtain (3.3.3) by noting that

$$(k_{h+1}-k_h)^2 z_{h+1} = (k_{h+1}-k_h)^2 \mu_{h+1}^{2/\bar{r}}(k_{h+1}, \rho_{h+1}, \tau_{h+1})$$

$$\leqslant \| u^{k_h}\xi_h \|^2_{q,r,Q_{\bar{\rho}_h,(t_0-\tau_{h+1},t_0)}} \leqslant \beta^2 \mid u^{k_h}\xi_h \mid^2_{Q_{\bar{\rho}_h,(t_0-\tau_{h+1},t_0)}}.$$

We estimate as before.

Inequalities (3.3.2′) and (3.3.3′) follow analogously.

COROLLARY 3.4. *Let $u(x,t)$ be a solution of* (2.1) *which satisfies inequality* (2.2.2). *Then*

$$\operatorname{mes} A_{k,\rho}(t) \leqslant \beta(t)/k^2, \tag{3.4.0}$$

and

$$\operatorname{mes} A_{\bar{k},\rho}(t) \leqslant \beta(t)/k^2. \tag{3.4.0′}$$

Proof of Corollary 3.4.

$$\beta(t) = \int_\Omega u^2(x,t)\,dx = \int_{\Omega\setminus A_{k,\rho}(t)} u^2(x,t)\,dx + \int_{A_{k,\rho}(t)} u^2(x,t)\,dx$$

$$\geqslant \int_{A_{k,\rho}(t)} u^2(x,t)\,dx \geqslant k^2 \operatorname{mes} A_{k,\rho}(t).$$

Similarly we may prove (3.4.0′).

THEOREM 3.5. *Let $u(x,t)$ be a solution of* (2.1) *satisfying hypotheses* (2.1.1–2.1.6) *and assumption* (2.2.1). *Then for τ and t_0 sufficiently small and large respectively the following estimate holds.*

$$\| u \|_{L^\infty Q_{\rho/2,(t_0-\tau/2,t_0)}\cap Q_T} \leqslant 2(\beta(t_0 - \tfrac{3}{4}\tau))^{\nu/2}, \tag{3.5.0}$$

where $\nu \in (0, 2(1+\chi)/(\bar{r}+2(1+\chi)))$.

Proof of Theorem 3.5. We choose τ sufficiently small so that inequalities (3.1.0) and (3.1.0′) hold. The recursion relationships (3.3.2–3.3.3) and (3.3.2′–3.3.3′) hold by Lemma 3.3. Setting

$$b = 2^4, \qquad c = \gamma_1((2+k^2)/k^{2\delta})\,(1+(1/\tau))\,(1/\rho^2),$$

$$\delta = 2/(n+2), \qquad \text{and} \qquad \epsilon = \chi$$

in (3.2.0) and (3.2.1), we obtain the value of λ defined by (3.2.2) by

$$\lambda = \min\{[2\gamma_1((2+k^2)/k^{2\delta})]^{-1/\delta}\,(1+(1/\tau))^{-1/\delta}$$

$$\times\, 16^{-1/\delta d}, [(2\gamma_1/\rho^2)\,((2+k^2)/k^{2\delta})]^{-((1+x)/x)}\,(1+(1/\tau))^{-((1+x)/x)}\,16^{-1/\delta d}\}.$$

Fixing ρ and choosing τ small, we can consider λ as a function of k, $\lambda = \lambda(k^2)$. Now for k^2 small, $\lambda(k^2) = O(k^2)$. Setting $k^2 = (\beta(t_0 - \frac{3}{4}\tau)^\nu$, we have $\lambda(k^2) = O(\beta(t_0 - \frac{3}{4}\tau)^\nu)$. We wish to choose

$$y_0 = \int_{t_0-\frac{3}{4}\tau}^{t_0} \int_{A_{k,\frac{3}{4}\rho}(t)} (u-k)^2 \, dx \, dt \leqslant \lambda,$$

and

$$z_0 = \int_{t_0-\frac{3}{4}\tau}^{t_0} \operatorname{mes}^{2/\hat{r}} A_{k,\frac{3}{4}\rho}(t) \leqslant \lambda^{1/(1+\chi)}.$$

From Corollary 3.4, we can show that

$$\operatorname{mes} A_{k,\frac{3}{4}\rho}(t) \leqslant \beta(t)^{1-\nu}.$$

Therefore, we can show that z_0 as a function of k^2 satisfies

$$z(k^2) = O(\beta(t_0 - \tfrac{3}{4}\tau)^{(2/\hat{r})(1-\nu)}).$$

Now, $\lambda(k^2)^{(1/(1+\chi))}$ is $O(\beta(t_0 - \frac{3}{4}\tau)^{\nu/(1+\chi)})$ is of lower order than $z_0(k^2) = \beta(t_0 - \frac{3}{4}\tau)^{(2/\hat{r})(1-\nu)}$, if we choose $\nu \in (0, 2(1+\chi)/(r + 2(1+\chi)))$. Then for t_0 sufficiently large, $\beta(t_0 - \frac{3}{4}\tau)$ is sufficiently small such that $z_0 < \lambda^{1/(1+\chi)}$.

Since

$$y_0 = \int_{t_0-\frac{3}{4}\tau}^{t_0} \int_{A_{k,\frac{3}{4}\rho}(t)} (u-k)^2 \, dx \, dt \leqslant \int_{t_0-\frac{3}{4}\tau}^{t_0} \int_{\Omega} u^2 \, dx \, dt$$
$$\leqslant \beta(t - (3\tau/4)) \, \tau,$$

we can show that $y_0 \leqslant \lambda$ for t_0 sufficiently large.

Now $y_h, z_h \to 0$ as $h \to \infty$. In particular, $\int_{t_0-\tau/2}^{t_0} \operatorname{mes} T_{2k,\rho/2}(t)\, dt = 0$. Therefore

$$u(x,t) \leqslant 2k = 2\beta(t_0 - \tfrac{3}{4}\tau)^{\nu/2} \qquad \text{a.e. in} \qquad Q_{\rho/2,(t_0-\tau/2,t_0)} \cap Q_T.$$

By similar reasoning, we can show that

$$\int_{t_0-\tau/2}^{t_0} \operatorname{mes} A_{\overline{2k},\rho/2}(t)\, dt = 0.$$

Therefore

$$-2\beta(t_0 - \tfrac{3}{4}\tau)^{\nu/2} = -2k \leqslant u(x,t) \qquad \text{a.e. in} \qquad Q_{\rho/2,(t_0-\tau/2,t_0)} \cap Q_T.$$

Proof of Theorem 2.2. Since $\bar{\Omega}$ is compact, we can find a finite number of cylinders $Q^i_{\rho,(t_0-\tau/2,t_0)}$,

$$Q^i_{\rho(t_0-\tau/2,t_0)} = \{| x_0{}^i - x | < \rho, t_0 - \tau/2 < t < t_0\}$$

such that $\bigcup_{i=1}^{m} Q^i_{\rho/2,(t_0-\tau/2,t_0)} \cap Q_{(t_0-\tau/2,t_0)}$. Therefore, by Theorem 3.5,

$$\| u \|_{L^\infty Q_{(t_0-\tau/2,t_0)} \cap Q_T} \leqslant 2m\beta(t_0 - \tfrac{3}{4}\tau)^{\nu/2}.$$

Since t_0 is only chosen sufficiently large, then

$$\| u \|_{L^\infty Q_{(t_0-\tau/2,\infty)} \cap Q_T} \leqslant 2m\beta(t_0 - \tfrac{3}{4}\tau)^{\nu/2}.$$

Since $\beta(t) \to 0$ as $t \to \infty$, we infer that $u(x, t) \to 0$ as $t \to \infty$.

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